

Klaus Döring, Bernhard Herzhoff, Georg Wöhrle (Hg.)

Antike Naturwissenschaft und ihre Rezeption

Band 7

 **Wissenschaftlicher Verlag Trier**

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Antike Naturwissenschaft und ihre Rezeption. -

Trier: WVT Wissenschaftlicher Verlag Trier.

Erscheint jährl. - Früher im Colibri-Verl., Bamberg. -

Aufnahme nach Bd. 5 (1995)

ISSN 0942-0398

Bd. 5 (1995) -

Verl.-Wechsel-Anzeige

*Gedruckt mit Unterstützung der
Karl- und Gertrud Abel-Stiftung*

© WVT Wissenschaftlicher Verlag Trier, 1997

ISBN 3-88476-265-6

ISSN 0942-0398

Titelvignette: Sich in den Schwanz beißende Schlange aus dem

Codex Marcianus Graecus 299 (= 584), fol. 188v;

Bibliotheca Marciana, Venedig

Umschlaggestaltung: Brigitta Disseldorf

(M. Nottar, Agentur für Werbung & Design, Konz)

Alle Rechte vorbehalten

Nachdruck oder Vervielfältigung nur mit

ausdrücklicher Genehmigung des Verlags

WVT Wissenschaftlicher Verlag Trier

Bergstraße 27, 54295 Trier

Postfach 4005, 54230 Trier

Tel. (0651) 41503, Fax 41504

Jens Høyrup (Roskilde)

Hero, Ps.-Hero, and Near Eastern Practical Geometry

An investigation of *Metrica*, *Geometrica*, and other treatises

In memory of Alistair Crombie
whose *Augustine to Galileo* first
introduced me to medieval science

In the following I intend to show, firstly, that Hero's geometry depends to a greater extent than usually assumed on Near Eastern practical geometry or its descendant traditions in the classical world, and that the conventional image (also suggested by Hero himself in the initial passage of *Dioptra*¹) as the transformer of theoretical into applied mathematics is only a half-truth; secondly, that much of what is shared by Hero's *Metrica* and the pseudo-Heronian collections assembled by Heiberg as *Geometrica* are shared borrowings from the same tradition, and that it is misguided to speak of *Geometrica* as „essentially” another version of *Metrica* I.² Concomitantly I shall argue that it is already misguided to speak of *Geometrica* as one treatise unless this be understood explicitly as Heiberg's construction.

A Near Eastern practitioners' tradition

It is of course essential for this argument that there *was* such a practitioners' tradition; that Hero and the compilers of the constituent manuscripts of *Geometrica* had access to it; and that we can find its representatives and show that their material is not borrowed from Hero.

This is a project of its own, which I have dealt with in other connections;³ here I shall only point to a few sources that prove the existence of a long-lived

-
- 1 Herons von Alexandria *Vermessungslehre und Dioptra*, ed. trans. Hermann Schöne, 188. (Heronis Alexandrini Opera quae supersunt omnia, vol. III., Leipzig 1903).
 - 2 Michael S. Mahoney, „Hero of Alexandria: Mathematics.” *Dictionary of Scientific Biography*. vol. VI, 314f., here 315. (New York 1972).
 - 3 For instance in „‘The Four Sides and the Area.’ Oblique Light on the Prehistory of Algebra”, in Ronald Calinger (ed.), *Vita mathematica: Historical Research and Integration with Teaching*, 45–65 (Washington 1996 [Marred by printing errors due to omitted proofreading.]).

tradition not dependent on the literate level of Greek geometry, going back at least to c. 2000 BC and still influential in the Islamic Middle Ages:

The first source to be quoted is an Old Babylonian⁴ quasi-algebraic⁵ problem, deliberately held in archaizing non-school formulations:⁶

- [If somebody asks you thus about] a surface: the four fronts and the surface I have accumulated, $41^{\circ}40''$.
 4, the four fronts, you inscribe. The reciprocal of 4 is $15'$.
 $15'$ to $41^{\circ}40''$ you raise: $10^{\circ}25''$ you inscribe.
 1, the projection, you append: $1^{\circ}10'25''$ makes $1^{\circ}5'$ equilateral.
 1, the projection, which you have appended, you tear out: $5'$ to two you repeat: $10'$ nindan confronts itself.

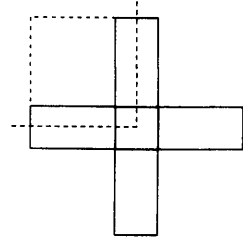


Figure 1 The procedure of BM 13901 N° 23.

Numbers translate the Babylonian sexagesimal place value system according to Thureau-Dangin's system, where $'$, $''$ etc. render decreasing and $^{\circ}$, $^{\prime\prime}$ etc. increasing sexagesimal order of magnitude, and where $^{\circ}$ is used when necessary

-
- 4 The Old Babylonian period goes from 2000 BC to 1600 BC; the extant mathematical texts belong to its second half. The present text may be from the 18th century BC, i.e., Hammurapi's times.
- 5 „Algebraic” because the method is analytical: the unknown is treated as if it were a normal identifiable quantity, and then extricated from the complex relationship in which it is originally involved; „quasi” because the method is not arithmetical as in Modern algebra but a „naive” cut-and-paste geometry, where the correctness of the steps is immediately seen but not argued. The present problem, moreover, operates directly on the unknown quantities themselves; the Old Babylonian school, however, also used the technique to solve non-geometric problems, using its line segments to *represent* areas, prices and numbers, as medieval and later applied algebra makes numbers represent geometric or physical magnitudes, prices, etc.
- 6 BM 13901 N° 23, first published in F. Thureau-Dangin, „L'Équation du deuxième degré dans la mathématique babylonienne d'après une tablette inédite du British Museum.” *Revue d'Assyriologie* 33 (1936), 27–48. The translation is mine, as are all translations in the following when nothing else is stated. In the present case, the translation differs radically from those of Thureau-Dangin and Neugebauer; the need for this revised translation is explained, e.g., in Høyrup, „Algebra and Naive Geometry. An Investigation of Some Basic Aspects of Old Babylonian Mathematical Thought.” *Altorientalische Forschungen* 17 (1990), 27–69, 262–354 – as regards the present text, see 271f.

to render „order zero” ($15'$ thus stands for $15/60$ and $1^\circ 5'$ for $1+5/60$). Figure 1 shows what goes on: The central square represents the area, while the four „wings” that „project” from the central square – rectangles whose width coincides with the „front” or side of the square, and whose length is 1 – represent the four fronts. One fourth of this configuration is a gnomon with area $10'25''$, which is completed as a square when the „projection 1” (representing the square, as the Babylonian *mithartum* and the Greek $\delta\acute{\upsilon}\nu\alpha\mu\iota\varsigma$ represent the square configuration of which they are the defining side⁷). The resulting area $1^\circ 10'25''$ has the squaring side (corresponding to the Greek $\pi\lambda\epsilon\upsilon\rho\acute{\alpha}$ $\tau\epsilon\tau\rho\alpha\gamma\omega\nu\iota\kappa\acute{\eta}$) $1^\circ 5'$, from which the projection is removed, leaving $5'$ as half the front; $5'$, when „repeated to two”, gives $10'$ *nindan* as the front (the *nindan* or „rod” of c. 6 m is the basic length unit).

In the *Liber mensurationum* – written in Arabic by an otherwise unidentified Abū Bakr, probably from around 800 AD and translated into Latin shortly after 1150 by Gerard of Cremona⁸ – the following problem is found:

And if somebody has said to you: Concerning a square, I have aggregated its four sides and its area, and what resulted was 140, then how much is each side?

The working in this will be that you halve the sides which will be two, thus multiply this by itself and 4 result, which you add to 140 and what results will be 144, whose root you take which is 12, from which you subtract the half of 4, what thus remains is the side which is 10.

Even here, a geometric procedure is involved; since it is described quite clearly in a problem from *Geometrica* to be quoted presently, I shall omit analysis.

In the present millennium, the problem turns up in Savasorda's early twelfth-century *Liber embadorum*:⁹

7 This parallel (and possible calque) is the topic of Høyrup, „*Dýnamis*, the Babylonians, and Theaetetus 147C7—148D7.” *Historia Mathematica* 17 (1990), 201–222.

8 Critical edition in H. L. L. Busard, „L'algèbre au moyen âge: Le 'Liber mensurationum' d'Abū Bekr.” *Journal des Savants*, Avril-Juin 1968, 65–125. Discussion, including tentative dating in Høyrup, „Al-Khwārizmī, Ibn Turk, and the Liber Mensurationum: on the Origins of Islamic Algebra.” *Erdem* 2 (Ankara 1986), 445–484.

9 II.12, ed. Maximilian Curtze, *Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance*, Vol. I, 36. (Abhandlungen zur Geschichte der mathematischen Wissenschaften, vol. 12–13. Leipzig 1902).

If, in some square, when its surface is added to its four sides, you find 77, how many cubits are contained in the surface? Taking the half of its sides, which is two, and multiplying it with itself, you find 4. If you add this to the given quantity, you will have 81, whose root (which is 9) you take; and when you subtract from this the half of the addition that was mentioned already, 7 remain. This is the side of the square in question, whose surface contains 49.

Further, in Leonardo Fibonacci's *Practica geometrie* from 1220:¹⁰

And if the surface and the four sides [of a square] make 140, and you want to separate the sides from the surface.

in Piero della Francesca's *Trattato d'abaco*:¹¹

And there is a square whose surface, joined to its four sides, makes 140. I ask what is its side.

and finally in Luca Pacioli's *Summa de arithmetica*:¹²

And if the 4 sides of a square with the area of the said square are 140. And you want to know how much is the side of the said square. ...

As we see, there is some variation. Savasorda's side is 7, but the others have 10, as the Old Babylonian problem (there transposed into the order of minutes, as mostly in Old Babylonian quasi-algebraic texts); Fibonacci normalizes the order of the members, but Pacioli corrects Fibonacci (whom he follows closely on other accounts) and reintroduces the original unusual order; Fibonacci thus cannot be his only source (and other corrections show that he is not just using Gherardo's version of the *Liber mensurationum*).

It should be obvious that some sort of continuous tradition is involved. It has also left its footprint in a treatise which Heiberg included in his *Geometrica* (cf. Figure 2):¹³

10 Ed. Baldassare Boncompagni, *Scritti di Leonardo Pisano matematico del secolo decimoterzo. II. Practica geometriae et Opusculi*, 59. Roma 1862.

11 Piero della Francesca, *Trattato d'abaco*, Dal codice ashburnhamiano 280 (359^r-291^r) della Biblioteca Medicea Laurenziana di Firenze. A cura e con introduzione di Gino Arrighi, p. 122. (Testimonianze di storia della scienza, 6. Pisa 1970).

12 Luca Pacioli, *Summa de Arithmetica geometria Proportioni: et proportionalita*, II, fol. 15r. (²Toscolano 1523; ¹1494).

13 *Geometrica* 24.3, in Heronis *Definitiones cum variis collectionibus*. Heronis quae feruntur *Geometrica*, ed. J. L. Heiberg, 418. (Heronis Alexandrini Opera quae

A square surface having the area together with the perimeter of 896 feet. To get separated the area and the perimeter. I do like this: In general [i.e., independently of the parameter 896 – JH], place outside (ἐκτίθημι) the 4 units, whose half becomes 2 feet. Putting this on top of itself becomes 4. Putting together just this with the 896 becomes 900, whose squaring side becomes 30 feet. I have taken away underneath (ὕφαρρέω) the half, 2 feet are left. The remainder becomes 28 feet. So the area is 784 feet, and let the perimeter be 112 feet. Putting together just all this becomes 896 feet. Let the area with the perimeter be that much, 896 feet.

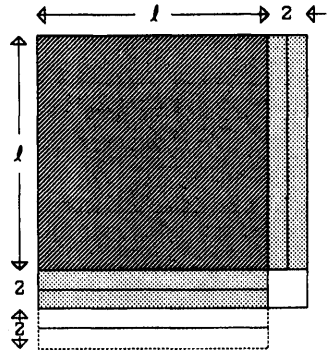


Figure 2 The procedure described in *Geometrica* 24.3.

„In general”/ καθολικῶς corresponds to *semper* in Abū Bakr’s treatise, and indicates that the step undertaken does not depend on the number 896 (but certainly on the fact that precisely one area and one perimeter are involved). „To get separated”/ διαχωρίσαι corresponds to *berûm* in Old Babylonian texts of similar riddle character (adding, e.g., workers, days, and the bricks produced), and to Leonardo’s *separare*.

These problems, of course, are not „practical”. They are riddles, elements of that cultural superstructure by which the craft of practical geometers upheld its professional identity – problems that allowed the practitioner to prove himself „a keen and swift enquirer”, in Savasorda’s words.¹⁴ Such riddles are found wherever a community of practical reckoners exists, and always connected to the kind of reckoning in which they are engaged but always (which is what allows us to speak of a „cultural” superstructure) somehow oblique in their relation to genuine practical problems. In eighteenth-century naval officers’ notebooks, they deal with navigation problems that could never present themselves in real life;¹⁵ nineteenth-century engineers, well trained in technical drawing and descriptive geometry, constituted an eager public for the triangle

supersunt omnia IV., Leipzig 1912). Heiberg’s translation and notes to the problem are misleading, imputing his own faulty understanding on the ancient copyist.

14 *Liber embadorum* II.7, ed. Curtze, 32 (cit. 9).

15 Exemplified in Henrik Meyer & Jan A. van Maanen, „An Anonymous Danish Mathematical Manuscript in Leiden University Library and its Author: Christian Høyer (1758–1809).” *Overdruk. Nieuw Archief voor Wiskunde*, 4. Serie 9 (1991), 351–357.

geometry of *Journal des mathématiques élémentaires*.¹⁶ Al-Khwārizmi characterized such problems as „brilliant” (*latif*), and the Arabic tradition in general referred to them as „rarities” (*nawādir*). In the perspective of the modern mathematical institution they have become „recreational”.

„The four sides and the area” is one among several quasi-algebraic riddles circulating among non-scholastic practical geometers in the earliest second millennium BC.¹⁷ The Old Babylonian school was inspired by them and developed them into a whole discipline of „naive-geometric algebra” with freely varied coefficients, using lines and areas to *represent* entities of other kinds – prices, pure numbers. The practitioners’ riddles, on the other hand, were always „natural”, referring to the single area, *the side* or *all four sides* of a square, etc. Another riddle of the kind asks to find the diameter, the perimeter and the area of a circle when their sum is given – relevant in the present connection because it is also found in several of the manuscript traditions aggregated in *Geometrica*, and because it is often cited as evidence that (and how) Hero dealt with mixed determinate second-degree equations.

Geometrica

That this use of the text is unwarranted follows already if we recall what Heiberg wrote about *Geometrica* and the two manuscript groups:¹⁸

- (1) *Geometrica* „was not made by Hero, nor can a Heronian work be reconstructed by removing a larger or smaller number of interpolations” (p. xxi).
- (2) Mss AC represent a book which, with additions, changes and omissions, only reached the present shape in Byzantine times; it was not meant to serve field mensuration directly but was for use „in [commercial and legal] life” and in general education (p. xxi).
- (3) Manuscript S, with the closely related ms V, was intended to serve youth studying „architecture, mechanics and field mensuration” in the „University of Constantinople” and thus „more familiar with theoretical mathematics” – a use which in Heiberg’s

¹⁶ Eduardo L. Ortiz, personal communication.

¹⁷ Since the culture of this practitioners’ environment was oral or semi-oral, the evidence for the existence of these riddles is indirect though fairly rich, as discussed in my „Four sides” (*cit.* 3).

¹⁸ J. L. Heiberg, (ed., trans), *Heronis quae feruntur Stereometrica et De mensuris*, xxi. (Heronis Alexandrini Opera quae supersunt omnia V., Leipzig 1914).

view agrees with the presence of Hero's (more or less) genuine *Metrica* in the same manuscript (p. xxiii).

- (4) Both versions of the work merge (each in its own way) „various problem collections together with Heronian and Euclidean excerpts” (p. xxiv).

One may wonder whether it was really a sound editorial method to fuse the two „versions” into one. Since Heiberg took over the task after Wilhelm Schmidt, the choice may not have been his. As things are, only very careful observation or reading of the Latin preface to *volume V* of the *Opera omnia* will reveal that a work contained in *volume IV* is a modern conglomerate of two (indeed more) ancient conglomerates. But Heiberg knew, and he *does* tell. He also seems to have known, but does not say it too directly, that the origin of the „problem collections” was neither Heronian nor Euclidean.¹⁹

Levels of comparison

Comparison between ancient mathematical texts (in the present case, texts representing the practitioners' tradition with the Heronian and pseudo-Heronian texts) can be made on at least five levels:

- (1) The overall organization of the material, reflecting the writer's sense of „natural progression”.
- (2) The methods seen from the stance of modern mathematics, supposed to reveal the *mathematics* involved.
- (3) The procedures in their actual detail – we are in a world where multiplying an area by 11 and then taking the 14th part was „another way”, ἄλλωϑ, with regard to the subtraction of $\frac{1}{7}$ and $\frac{1}{14}$ from the area (*Geometrica* 17.15, mss AC).
- (4) The numerical values involved – as we have seen, the solution to the „four sides and the area” remained 10 for 3200 years; that successive doublings are exactly 30 in number was the compulsory standard from 1800 BC until the earlier Middle Ages, when 64 arose as an alternative possibility.
- (5) The phraseology; even on this point, indeed, the conservatism of our tradition is amazing to a modern mind.

¹⁹ He may not have noticed that his chapters 22 and 24, found in ms S but there squeezed in between *Stereometrica* and *Metrica*, are independent treatises. Cf. note 29.

The organization

Let us first look at the global organization of the subject-matter.

For two reasons, the cuneiform record tells us little about the standard progression of the early practitioners: firstly, what we find in the clay tablets has mostly already gone through a process of scholastic normalization; secondly, those texts that are so extensive that they *might* tell a progression through themes are instead either rather disorganized „anthology texts” or organized around a particular theme. The only exceptions – the so-called Tell Harmal compendium²⁰ and the text BM 80209,²¹ both belonging to text groups close to the practitioners’ tradition – tell us that squares precede rectangles, which on their part precede circles.

The Arabic Middle Ages, on the other hand, present us with a whole array of treatises still close to the tradition – thus Abū Bakr’s *Liber mensurationum*, al-Karajī’s *Essentials of reckoning*,²² and ibn Thabāt’s *Reckoner’s Wealth* (c. 1200 AD).²³ Al-Khwārizmī’s chapter on geometry in the *Algebra*,²⁴ Savasorda’s *Liber embadorum*, and Leonardo Fibonacci’s *Pratica geometrie* are *a priori* suspicious because of the explicit orientation of their authors (Euclidean in the latter two cases, idiosyncratically Al-Khwārizmī’s own in the former) – but as we shall see, their stylistic efforts are not always reflected in what they bring forth. The treatises in question arrange the material on plane geometry as follows:²⁵

20 Albrecht Goetze, „A Mathematical Compendium from Tell Harmal.” *Sumer* 7 (1951), 126–155.

21 Jöran Friberg, „Methods and Traditions of Babylonian Mathematics, II: An Old Babylonian Catalogue Text with Equations for Squares and Circles.” *Journal of Cuneiform Studies* 33 (1981), 57–64.

22 Adolph Hochheim (ed., trans.), *Kāfi fil Hisāb (Genügendes über Arithmetik)* des Abu Bekr Muhammed ben Alhusein Alkarkhi. I–III. (Halle 1878).

23 Ulrich Rebstock, *Die Reichtümer der Rechner (Ġunyat al-Hussāb) von Aḥmad b. Tabāt (gest. 631/1234). Die Araber – Vorläufer der Rechenkunst.* (Beiträge zur Sprach- und Kulturgeschichte des Orients, 32. Walldorf-Hessen 1993).

24 Solomon Gandz, (ed.), „The Mishnat ha Middot, the First Hebrew Geometry of about 150 C.E., and the Geometry of Muhammad ibn Musa al-Khowarizmi, the First Arabic Geometry <c. 820>.” *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik. Abteilung A: Quellen* 2 (1932).

25 These are rough outlines, and make some of the works look more orderly than they are – in particularly those that have gone through the Byzantine mill.

Abū Bakr, *Liber mensurationum*

Square; rectangle; rhomb; trapezia (isosceles / acute-angled / right / obtuse-angled).

Triangles (equilateral / isosceles / acute-angled / right / obtuse-angled).

Circle; semicircle; circular segment (>semicircle); circular segment (<semicircle).

Al-Karajī, *Kaḥḥi fi'l-hisāb* („Essentials of Reckoning”)

Square and rectangle; rhomb; parallelogram; trapezia.)

Triangles (right / equilateral / isosceles, acute-angled / scalene, acute-angled / isosceles, obtuse-angled / scalene, obtuse-angled).

Circle; segments (semicircle / major / minor).

Regular n-gons (ex. $n = 6$)

Ibn Thabāt, *Ghunyah al-Hussāb* („Reckoner’s Wealth”)

Square and rectangle; rhomb; parallelogram; trapezia (right / isosceles / scalene, acute-angled); other quadrangles (with „surveyors’ formula”); „staircase-” and „drum-form” figures.

Regular n-gons, $n > 4$ (ex. $n = 6$).

Acute-angled triangles (equilateral / isosceles / scalene); right triangles (isosceles / scalene); obtuse-angled triangles (isosceles / scalene) – rūmī-method (= Heronic Formula).

Circle (incl. diameter+perimeter+area-“algebra”); circular sector; segments (semicircle / major / minor) – „egg-shaped” (doubled minor segment).

Figure within figure, including concentric squares.

Al-Khwārizmī, *Algebra* (chapter on geometry):

Square; equilateral triangle; rhomb; circle; circular segment; volumes. Pythagorean theorem.

Square; rectangle; rhomb; parallelogram; irregular quadrangle.

Triangles (right; acute-angled; obtuse-angled).

Circle.

Frustum; square inscribed in isosceles triangle.

Savasorda, *Liber embadorum*:

Square; rectangle; rhomb; „quasi-algebra” on squares and rectangles.

Triangles (equilateral / isosceles / acute-angled / right / obtuse-angled); parallelogram; trapezia (isosceles / acute-angled / right / obtuse-angled); irregular quadrangles.

Circle; semicircle; segments (minor / major); ellipse; chord table.

Regular polygons; fields on mountain slopes.

Leonardo Fibonacci, *Pratica Geometrie*:

Square; rectangle.

Triangles (equilateral; isosceles; acute-angled; obtuse-angled).

Square; rectangle; rhomb; parallelogram; trapezia (right / isosceles / acute-angled / obtuse-angled); irregular quadrangles; polygons, primarily regular.

Circle; semicircle; chords with table; sectors, segments, „egg-” and other mixed shapes; fields on mountain slopes.

It is natural that any treatise on mensuration begins with the square – more precisely with the unit square, which provides the area unit; going on with rectangles is almost as natural, since rectangles provide the basis for area computation. After that point, a mind trained in the Euclidean tradition will tend to go on with triangles, from there to quadrangles of increasing irregularity, etc. There would be no reason to treat the semicircle as a particular configuration, but the use of approximations may ask for separate treatment of minor and major circular segments. This sequence corresponds to what we see in the beginning of Al-Khwārizmī's and Fibonacci's treatises (italicized above; Fibonacci goes no further than the triangle). Both, however, have a second start from the square, and then follow a sequence that *grosso modo* coincides with what we find in Abū Bakr, al-Karajī and ibn Thabāt, representing the „normal progression” of themes as seen by the practitioners:

After squares and rectangles (both with quasi-algebraic problems) follow rhombs, at times treated at great length with trivial variations of the quasi-algebraic problems on rectangles. Parallelograms may come next, but much more important is the treatment of trapezia. Then irregular quadrangles may follow, either split into triangles or treated by means of the „surveyors' formula”, average length times average width (always yielding a result which is too great, except for rectangles where it is trivially correct; avoided by the „mathematicians” but used by ibn Thabāt, and mentioned by other Arabic authors as a practitioners' formula). If included, regular polygons may follow next.

Only then come the various triangles, followed by circle, semicircle, segments (Abū Bakr, al-Karajī and ibn Thabāt treat the major segment first, all others have it last). If not dealt with after quadrangles, regular polygons may come after circles and segments.

A corresponding analysis of *Metrica* gives the following result;

Unit square; rectangles.

Triangles (right / isosceles / acute-angled / obtuse-angled do. / [Heronic formula] / with irrational height)

Trapezia (right / isosceles / acute-angled / obtuse-angled) / irregular quadrangles.

– regular n -gons, $2 < n < 13$.

– circle; 2 concentric circles; segments (minor / major).

– ellipse; parabola.

– cylindrical surface; conic surface; spherical surface; segment of this.

The order of *Geometrical*/AC is as follows:²⁶

Metrology; square; rectangle.

Triangles (right / isosceles / acute-angled / obtuse-angled / „Heronic formula” / obtuse-angled, continued / isosceles with inscribed square).

Rhombus; rectangle; various triangles; parallelograms; trapezia (right / isosceles / acute-angled / obtuse-angled); irregular quadrangles (Heronic formula, and triangulation).

Circle (incl. 1.-degree „algebra”); semicircle; segments (minor / major); 2 concentric circles; circles from „another book” (including diameter+perimeter+area-“algebra”).

Regular n -gons, $4 < n < 13$; irregular polygons; ...

The *Geometrical*-part of ms S exhibits this structure:²⁷

Metrology; square; rectangle; irregular quadrangle (surveyors' formula).

Triangles (right / isosceles); circles inscribed in and circumscribed around isosceles and scalene (13-14-15) triangle.

Circle; semicircle;²⁸ segments (major / minor / major by subtraction / minor).

Then follows *Stereometrical*, after which come two separate treatises that Heiberg has inserted in *Geometrical*²⁹. The first (inserted as 24.1–51; henceforth thus S:24) contains, among other things, determinate and indeterminate problems about geometrical configurations, but also the section on circles inscribed in and circumscribed around triangles which already entered after triangles.

26 This is the order of Heiberg's edition, which follows ms A apart from an inversion of 15.17–19 (first in the ms) and 15.15–16 (all five sections deal with parallelograms, but the wording suggest that section 15 and not section 17 be first). I omit the various metamathematical discussions contained in the work (the origin of geometry, etc.), as irrelevant for the present purpose.

27 On the whole, ms V depends so strongly on ms S that S can be taken to represent the better version.

28 After the semicircle follows the volume of a semicircular wall, in practice only a calculation of its base, moved by Heiberg to *Stereometrical*.

29 Probably because he had no other work in which to put them; they are not coherently in mss A+C (isolated pieces are e.g. the problem on circular diameter+perimeter+area, though in words that do not suggest use of precisely this treatise, cf. below); they should certainly not have been merged with neither the modern nor the ancient conglomerates. In references to *Geometrical*/S below they are hence not included. Instead, I shall refer to the treatise which Heiberg inserted as chapter 24 as S:24, and to the other as S:22.

According to the text edition, S:22.3–24 is also contained in ms A; however, as can be seen from the manuscript description of the preface (p. xi), it is not part of its *Geometrical*-text.

Here we find the problem on the square diameter+perimeter, and that of the circular area+diameter+perimeter.

The other (22.1–24) contains, after a section on metrology, the areas of regular n -gons, $2 < n < 13$, followed by circle and semicircle, in abstract formulation (everything else in *Metrica* and either *Geometrica* is set forth through numerical examples).

For comparison, the section on mensuration from Columella's *De re rustica* V (a close contemporary of *Metrica*) may be of interest:³⁰

Metrology; square; rectangle; trapezium.

Triangles (equilateral / right).

Circle; semicircle; minor segment; regular hexagon.

Hero obviously deviates most clearly from the established pattern; he makes triangles follow directly upon the rectangle; he omits the semicircle (obviously considering it trivial), and replaces the „egg-shape” (if it was indeed part of the heritage, which Old Babylonian texts suggest) by a „scientific” figure – the ellipse.

Mss AC is not far from the Heronic pattern, but the semicircle appears as a configuration of its own. Ms S, on the other hand, shows closer affinities with the practitioners' tradition, as revealed both by the presence of the surveyors' formula and the place where it occurs. In all its brevity, Columella's treatment of the subject is also close to the traditional and well away from the scientific and Heronian order, and shows that the characteristic order of the medieval handbooks was not the result of a later transformation of a tradition derived from Hero.

Methods, considered abstractly

On this level, one feature is usually referred to as indubitably Heronian in *Geometrica* as well as *De re rustica*: The use of the Archimedean value for π . Strangely enough once again, since Hero tells us that the evidence is not valid. Two passages in *Metrica* are decisive:

- (1) I.xxx explains that „the ancients” – οἱ ἀρχαῖοι – measured the minor segment as $[\frac{1}{2}(c+h)] \cdot h$, where c is the chord and h the height. As he argues, these „ancients” seem to have „followed those who took the perimeter to encompass the triple of the diameter”; if $\pi = 3$, indeed, the formula is exact for the

30 Lucius Iunius Moderatus Columella, *Zwölf Bücher über Landwirtschaft*, ed. Will Richter. 3 vols, (München & Zürich 1981–83).

semicircle. The argument implies that Hero had no direct information about the provenience of the formula, but that he knew about the use of $\pi = 3$ (the traditional Babylonian value, reappearing in the Ancient Testament and hence probably current in the Syrian Near East, and also reappearing in Demotic mathematics).

- (2) I.xxxi tells that „those who made more precise investigations” add $\frac{1}{14}(\frac{6}{2})^2$, and argues with a similar argument that they must have followed the other course according to which the perimeter is the triple diameter and in addition $\frac{1}{7}$ of the diameter. Even here, we see, the use of the Archimedean value (transformed from the ratio 22:7 into the number $3\frac{1}{7}$) is already established practice, and has been so for so long that Hero does not find its traces; it has also generated an approximate formula which is even worse for small segments, in all probability found via fitting by practitioners with neither theoretical mathematical training nor propensity.³¹ The formula is *post-Archimedean*, but certainly not Archimedean.

The whole organization of the discussion of segments is clear evidence that Hero worked on the basis of an earlier treatise which he emended but did not rewrite consistently: The text includes the theorem that a segment is greater than $1\frac{1}{3}$ times the triangle with the same base and the same height, and suggests that this limit – the exact area of the corresponding parabolic segment, as the text tells with due reference to Archimedes – be used to approximate the area of segments whose chord is more than three times the height. A lemma and the first two theorems leading to this result open the whole treatment of circular segments (I.xxvii–xxix). Then the two received approximation formulae are given in I.xxx and I.xxxi; in the end of I.xxxi comes the demonstration that the formula used by „those who made more precise investigations” yields absurd results when the base is more than thrice the height. Quite appropriately, the rest of the proof begun in I.xxvii–xxix is given in I.xxxii – after which I.xxxiii goes on as if neither I.xxxii nor the final warning of I.xxxi had existed, and

31 The approximation is indeed so much worse that Hero’s reconstruction is almost certain; mathematical reflection would have shown (in modern terms) that the relative error goes to infinity when the arc goes to zero; the text does not make this inference, but it does show that the formula is impossible when the height is less than $\frac{1}{3}$ of the chord. The idea to add $\frac{1}{14}(\frac{6}{2})^2$ must come from fitting to a particular case – and probably to a semicircle with diameter 14.

applies the formula $A = [1/2(c+h)] \cdot h + 1/14(c/2)^2$ to a segment where it has just been shown not to apply, viz. $c = 4h$.

Instead of accusing Hero of inconsistent emendations, one might of course suggest that somebody else produced the text we know by adding the warning and the parabolic approximation to Hero's original text, which will then have contained nothing but the received approximations; without other arguments than a general stylistic impression I find this explanation of the textual inconsistencies unlikely. Somewhat more plausible is the possibility that Hero did not write the Archimedean approximation into the text but as a marginal scholion to a borrowed treatise – a scholion that was so long that he distributed it over several manuscript pages – and that a later copyist inserted the scholion in the running text.³²

32 A similar argument shows that the „Heronian formula” does not belong straightforwardly in the text but is an interpolation – either made by Hero in a source which he does not correct in details, by some later editor in Hero's text, or both in combination. I.vii proves a quasi-algebraic lemma („κατὰ ἀνάλωσιν”); the lemma tells that $\sqrt{c \square(a, b)} = c \square(a, b)$, if $\square(d)$ designates the square on d and $c \square(f, g)$ the rectangle contained by f and g .; I.viii goes on with a numerical example, introduced independently of I.vii; this example is followed by the sentence „The geometrical demonstration of this is the following”, after which it goes on with the words in which *Dioptra* treats the same matter – words which fit the *Dioptra* context but which constitute a pointless repetition in *Metrica*. The whole proof follows the *Dioptra* version almost verbatim, inserting a couple of extra explanations, changing a few formulations, and changing the order of letters designating lines in order to make it alphabetic (this alphabetic order prevails in *Metrica* I). The exposition in *Metrica* closes with the numerical example of *Dioptra*, which is different from its own initial example (replacing $\mu\omicron\iota\rho\alpha$ with $\mu\omicron\nu\acute{\alpha}\zeta$ as the name of the unit, another instance of stylistic adjustment) – and then goes on in I.ix as if I.vii–viii had not been there. That I.vii–viii is an interpolation is obvious. The confusion of numerical examples and the double introductions of I.vii and I.viii speaks against an ascription of the passage in its entirety to a single hand. If we remove the introduction of I.viii and the ensuing numerical example, the whole structure – an introductory lemma (here algebraic, there dealing with the sum of a geometric progression) followed by a geometric proof – is similar to that of I.xxvii–xxix,xxxii, the Archimedean approximation to the segment. The most likely explanation may be that Hero made both interpolations apart from the introduction and first numerical example of I.viii (either as marginal scholia or without smoothing the text consistently), following *Dioptra* closely in the demonstration but expanding a few difficult points; and that a later editor added the opening of I.viii and the first numerical example in order to motivate the ensuing proof. (For further analysis, see Høyrup, „Sulla posizione della ‘formula di Erone’ nei *Metrica* (con una nota platonica)”; to be published in *Bolletino di Storia delle Scienze Matematiche*). While the Heronian formula is thus in some way an interpolation in *Metrica*, there seems to be no reason to doubt that it belongs originally in *Dioptra* – cf. Schöne's

In all three cases, Hero's debt to the extra-theoretical tradition remains obvious. The presence of the Archimedean π in other treatises is thus no indication that they depend on Hero; and in so far as they do not mention the Archimedean approximation formula for segment areas (and neither Columella nor any *Geometrica* do so³³) they are more likely *not* to use Hero (unless of course the insertion is post-Heronian).

The procedures in their actual detail

Only procedures of a certain complexity allow so much variation that the question of dependence or independence can be discussed – it is difficult to discover more than one reasonable way to calculate a rectangular area. But the procedures used to find the heights of scalene triangles and trapezia do offer significant contrasts, and so do the ways the circle, the semicircle and the segments are dealt with. Heights first.

These are calculated from a side and its projection on the base by means of the Pythagorean theorem. For inner heights, three different formulae for the projections of the sides turn up in the material (see Figure 3): One is „Euclidean”, based on *Elements* II.13; one is „algebraic”, based on the principle of *Elements* II.8; one, finally, is a variant of the „algebraic” formula, replacing II.8 with II.6. All formulae are of course algebraically equivalent, but it is clear that the sources consider them different,³⁴ and that they were handed down consistently as different methods.

One is based on *Elements* II.13 (the „Extended Pythagorean theorem”),

$$\square(a)+2 \square(q,c) = \square(b)+\square(c),$$

whence

$$q = \frac{b^2 + c^2 - a^2}{2} \div c$$

rejection of Hultsch's contrary opinion in the preface, pp. xixf (*cit. 1*). Nor is there any serious reason to doubt that it was discovered within the Greek world, even though those Arabic writers may be mistaken who ascribe it to Archimedes – Hero, when we know for sure that he has reasons to do so, always gives credit to the great Syracusan. But neither *Dioptra* nor *Metrica* do so on this occasion.

33 Both, moreover, give the post-Archimedean formula for the minor segment in the impossible case $c = 4h$.

34 Thus very explicitly in Fibonacci's *Pratica geometrie* III.2, ed. Boncompagni, 34ff. (*cit. 10*).

The „main algebraic formula” makes use of semi-sum and semi-difference, as the quasi-algebraic tradition had always done:

$$\frac{q-p}{2} = \frac{b^2 - a^2}{2} + c, \quad \frac{q+p}{2} = \frac{c}{2}$$

whence

$$q = \frac{c}{2} + \frac{b^2 - a^2}{2} + c$$

$$p = \frac{c}{2} - \frac{b^2 - a^2}{2} + c$$

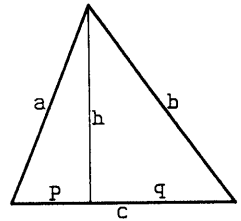


Figure 3

The probable argument behind this formula runs as follows (see Figure 3):

$$\square(b) - \square(a) = \{\square(q) + \square(h)\} - \{\square(p) + \square(h)\} = \square(q) - \square(p)$$

$\square(q) - \square(p)$, however, is the difference between two squares, most likely to be understood as the band between concentric squares (see Figure 4):

$$\square(q) - \square(p) = \Leftrightarrow \left(\frac{q-p}{2}, 2(q+p) \right) =$$

$$\Leftrightarrow \left(\frac{q-p}{2}, 2c \right)$$

(This argument, a „naive” version of *Elements* II.8, is found in ibn Thabāt’s treatise and in *Metrica* I.xxvi). Therefore,

$$\frac{q-p}{2} = \frac{b^2 - a^2}{2} + c$$

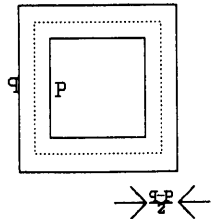


Figure 4

The „algebraic alternative” has an analogous proof, but presupposes *Elements* II.6 (or „proto-*Elements*-II.6”) instead of „proto-*Elements*-II.8” (see Figure 5):

$$p = \frac{1}{2} \left(c - \frac{b^2 - a^2}{c} \right)$$

since

$$\square(c, q-p) + \square(p) = \square(q)$$

whence

$$\square(c, q-p) = \square(q) - \square(p) = \square(b) - \square(a),$$

and since, moreover,

$$2p = c - (q-p).$$

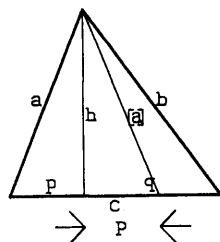


Figure 5

For the outer heights of an obtuse-angled triangle, two methods occur (see Figure 6). One is based on *Elements* II.12 („Extended Pythagorean theorem”):

$$\square(a) = \square(b) + \square(c) + 2\square(p, c)$$

whence

$$p = \frac{a^2 - b^2 - c^2}{2} + c$$

The other, analogous to the above „main algebraic formula”,

$$p = \frac{a^2 - b^2}{2} + c - \frac{c}{2}$$

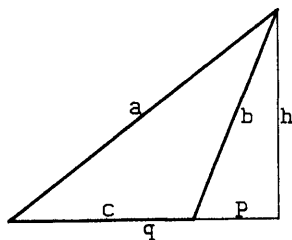


Figure 6

is only found in one case (*viz* in *Liber mensurationum*), anomalous also for other reasons and probably Abū Bakr’s own invention (apart from the equilateral triangle it is the only polygon in the work that is not composed from Pythagorean triangles).

The distribution of the different formulae can be summarized, using the following abbreviations:

Alg.: „Main algebraic formula” based on semi-sum and semi-difference and proto-*Elements* II.8

ExtP: *Elements* II.13 or 12 („Extended Pythagorean theorem”)

II.6: The variant of the „algebraic” method that refers to *Elements* II.6.

Abū Bakr, *Liber mensurationum*

Acute-angled trapezium: *Alg*; alternative, *ExtP*

Obtuse-angled trapezium: *Alg* (Abū Bakr's own construction)

Acute-Angled triangles: *Alg*; alternative, *ExtP*

Obtuse-angled triangles: Inner height *Alg*; outer heights *ExtP*

Al-Khwārizmī, *Algebra*

Acute-angled triangle: determination by *al-jabr*

Obtuse-angled triangle: Only one height exists, cross-reference

Al-Karajī, *Kāfī fī 'l-ḥisāb*

Acute-angled triangle: *Alg*

Obtuse-angled triangle: *ExtP*

ibn Thabāt, *Ghunyah al-Ḥussāb*

Acute-angled trapezium: *Alg*

Obtuse-angled trapezium: absent

Acute-angled triangle: *Alg*; alternative, *II.6*

Obtuse-angled triangle: *ExtP*

Savasorda, *Liber embadorum*

Acute-angled triangle: *ExtP*; alternative, *II.6*

Obtuse-angled triangle: *ExtP*

Acute-angled trapezium: *Alg*, with a mistaken cross-reference

Obtuse-angled triangle: *ExtP*

Leonardo Fibonacci, *Pratica geometrie*

Acute-angled triangle: *ExtP*; alternatives, *Alg* and *II.6*

Obtuse-angled triangle: *ExtP*; cross-references to *Alg* and *II.6*

Acute-angled trapezium: *Alg*

Obtuse-angled trapezium: *ExtP*

Hero, *Metrica*

Acute-angled triangles: *ExtP*

Obtuse-angled triangles; *ExtP*

Trapezia: Only a cross-reference

Geometrical/AC

Acute-angled triangle: *ExtP*

Obtuse-angled triangle: *ExtP*, alternative akin to *II.6*

Acute-angled trapezium: corrupt or erroneous

Obtuse-angled trapezium: cross-reference

Geometrical/S

Scalene triangles and trapezia are not treated

Heights are never computed in the Old Babylonian tablets, and the few Late Babylonian texts that have been found only calculate those of isosceles triangles

and trapezia. It would seem a natural guess that the determination of heights in scalene figures is a Greek invention. The material at hand, however, contradicts this assumption.

Hero, as we see, builds on Euclid, and *Geometrical/AC* does so, presenting the „algebraic” method in II.6 version as an alternative possibility. In the Arabic treatises, however, the situation is different – but only for internal heights. The determination of external heights follows *Elements* II.12; when finding internal heights, as a rule, the texts first state the „algebraic” solution, and then sometimes add the Euclidean method as an alternative. Savasorda, who tries to be primarily Euclidean when treating triangles, slips when he comes to the trapezium. So does Fibonacci.

The conclusion appears to be that the practical tradition knew the principle in „algebraic” form already before the Greeks, but that it had only applied it to internal heights, in agreement with al-Khwārizmī’s statement that an obtuse-angled triangle possesses only one height. The Greeks generalized the concept of a height to the perpendicular on a side *or its extension*, and both theorems went into *Elements* II in a formulation that brought them into connection with the Pythagorean theorem and proved them from II. 4 and II.7. One may even guess that the reason that II.12 (external heights) precedes II.13 (internal) is that it was new and hence more interesting. The practical tradition adopted the innovation and handed it down as faithfully as always, but kept the old way where it served, pointing however at times to the Euclidean method as an alternative way.

On this point we notice that Hero as well as *Geometrical/AC* follow the Euclidean way; but *Geometrical/AC* also shows that it knows the „algebraic” trick. *Geometrical/S*, in agreement with its use of the archaic surveyors’ formula, does not care about such sophisticated questions.

Circle computation may seem slippery ground – after all, everything that is involved are squarings, square roots and numerical factors. None the less, they turn out to be informative (diameter d , radius r , perimeter p , area A):

Abū Bakr, *Liber mensurationum*

$$A = d^2 - \frac{1}{7}d^2 - \frac{1}{2} \cdot \frac{1}{7}d^2; \text{ or } A = (\frac{1}{2}d) \cdot (\frac{1}{2}p)$$

$$p = 3\frac{1}{7}d; \text{ or } p = \sqrt{(A + \frac{3}{11}A)}$$

Al-Khwārizmī, *Algebra*

$$p = 3\frac{1}{7}d, \text{ („a convention among people without mathematical proof”)}$$

$$\text{(two Indian alternatives, } p = \sqrt{(10d^2)} \text{ and } p = \sqrt[62832]{20000d}$$

$$A = (\frac{1}{2}d) \cdot (\frac{1}{2}p)$$

Al-Karaji, *Kāfi fi'l-ḥisāb*

$$A = ({}^1/2d) \cdot ({}^1/2p); \text{ or } A = ({}^1/4d) \cdot p; \text{ or } A = d \cdot ({}^1/4p); \text{ or } A = d^2 - {}^1/7d^2 - {}^1/2 \cdot {}^1/7d^2; \text{ or } A = p^2 \div 12 {}^1/7$$

$$p = 3 {}^1/7 d$$

$$d = p \div 3 {}^1/7$$

Ibn Thabāt, *Ghunyah al-Hussāb*

$$A = ({}^1/2d) \cdot ({}^1/2p); \text{ or } A = ({}^1/4d) \cdot p; \text{ or } A = d \cdot ({}^1/4p); \text{ or } A = d^2 - {}^1/7d^2 - {}^1/2 \cdot {}^1/7d^2; \text{ or } A = p^2 \div 12 {}^1/7 \text{ or } {}^1/4p^2 \div 3 {}^1/7; \text{ or } A = {}^1/4 \cdot (d \cdot p)$$

$$p = 3 {}^1/7 d$$

$$d = p \div 3 {}^1/7$$

Savasorda, *Liber embadorum*

$$p = 3 {}^1/7 d$$

$$A = ({}^1/2d) \cdot ({}^1/2p), \text{ alternative } A = d^2 - {}^1/7d^2 - {}^1/2 \cdot {}^1/7d^2$$

(followed by a correction of the π -value, $\pi = {}^{377}/120$)

Old Babylonian texts

$$p = 3d; A = p^2 \cdot 5' \text{ (normal method)}$$

$$A = {}^1/4(d \cdot p)^{35}$$

Hero, *Metrica*

$$A = {}^1/14 \cdot 11d^2$$

$$p = {}^1/7 \cdot (22d)$$

$$d = {}^1/22 \cdot (7p)$$

$$2A = r \cdot p$$

Geometrical AC³⁶

$$A = {}^1/4(d \cdot p); \text{ or } A = ({}^1/2d) \cdot ({}^1/2p); \text{ or } A = {}^1/88 \cdot (7 \cdot p^2); \text{ or } A = {}^1/14 \cdot (11 \cdot d^2); \text{ or } A = d^2 - {}^1/7d^2 - {}^1/14d^2 \text{ („In Euclid“!)}$$

$$d = {}^1/22(7p); \text{ or } d = 7 \cdot ({}^1/22p)$$

$$p = 3d + {}^1/7d$$

35 Strictly speaking, this formula is only used for semicircles in the Old Babylonian texts – in the problem text BM 85210, rev. I 18, and in various coefficient lists.

36 This scheme covers only chapter 17. Chapter 21, which tells to be an insertion „taken from another book by Hero”, returns to the topic of circles, in ms C as follows: $p = {}^1/7 \cdot (22d)$; $d = {}^1/22(7p)$; $2A = p \cdot r$; „circulation” of a given area; separation of $d+p+A$; the corners between a circle and the circumscribed square. The first three formulae follow *Metrica* rather closely, using even the same verbs though not the same grammatical forms; throughout, as in *Metrica* but not in chapter 17, magnitudes are told to be n „units”/μονάδες; elsewhere, mss AC reckons explicitly or implicitly in „ropes” (σχοινία) or „fathoms” (ὀργυιᾶϊ), whereas mss C and C.24 have the „foot” as their all-pervasive standard unit. Ms A represents a rewritten version of the same material, which expands both the explanations and the number of formulae (adding, e.g., the non-Heronian $A = {}^1/4pd = d \cdot ({}^1/4p)$, but retains enough of the Heronian words to make us sure of the connection.

p and d from $p+d$; $p = 3d+^1/7d$; $d = ^1/3.(p-^1/22p)$; or $d = ^1/22.(7p)$; $A = ^1/88.(7.p^2)$; or $A = d^2-^1/7d^2-^1/14d^2$; or $A = ^1/14.(11.d^2)$; or $A = 3.(^1/2d)^2+^1/7.(^1/2d)^2$; or $A = (d^2-^1/4.d^2)+^1/21.(d^2-^1/4.d^2)$; or $4A = p.d$; or $A = (^1/2d).(^1/2p)$; or $A = (^1/4.p).d$; etc. (more complex numerical examples follow)

Geometricals

$$A = (11d^2) \div 14$$

$$p = (22d) \div 7$$

$$d = (7p) \div 22$$

$$p = 3d+^1/7d;$$

$$d = (p \div 22) \cdot 7;$$

$$A = ^1/14.(11d^2); \text{ or } A = ^1/4.(p.d); \text{ or } A = ^1/88.(7p^2); \text{ or [corrupt or deliberately misleading]}$$

Columella, De re rustica

$$A = ^1/14.(11d^2)$$

Here we should distinguish as carefully as our texts between $A = ^1/4.(d.p)$ and $A = ^d/2.p/2$, between taking $^1/n$ and dividing by n ($\mu\epsilon\rho\acute{\iota}\zeta\omega$), and between $^1/7.(22d)$, $(22d) \div 7$ and $3d+^1/7d$. The „orthodox” Archimedean formulae are those found in *Metrica*. „Those who made more precise investigations”, as we remember, took p to be $3d+^1/7d$ – and in all cases where this formula is used,³⁷ the formulation tells very clearly that the diameter is taken thrice, and calculated explicitly, after which a supplementary seventh is added. Everywhere, the expressions $\tau\rho\iota\sigma\acute{\alpha}\kappa\iota\varsigma$ and $\tau\rho\iota\pi\lambda\acute{\alpha}\sigma\iota\omicron\nu$ are used even when neighbouring multiplications are $\acute{\epsilon}\pi\grave{\iota} n$. This is quite striking. The Old Babylonian texts, indeed, find the perimeter as the diameter „repeated until three” or as „thrice” the diameter, not by the normal multiplication used, e.g., when A is found as 5 times p^2 ; and the same idea is still found in a design booklet from c. 1488, written by the master builder Mathes Roriczer – in a constructive formulation that betrays the probable reason for the survival of the particular wording³⁸. It seems beyond doubt that the anonymous practitioners who „made more precise investigations” belonged within the tradition which is already reflected in the Old Babylonian tablets, although Archimedes’s calculation caused them to add a supplementary

37 Mss AC, 17.8, 17.10, 17.29; S:22.16; and S:24.45; cf. ms A, 17.16, where the area is found as $3(^1/2d)^2+^1/7(^1/2d)^2$

38 „If anyone wishes to make a circular line straight, so that the straight line and the circular are the same length, then make three circles next to one another, and divide the first circle into seven equal parts”, one of which is marked out in continuation of the three circles – *Geometria deutsch*, 9, trans. Lon R. Shelby, in Shelby (ed.), *Gothic Design Techniques. The Fifteenth-Century Design Booklets of Mathes Roriczer and Hanns Schmuttermayer*, Carbondale & Edwardsville 1977, 121.

seventh;³⁹ in the Arabic writings the characteristic formulation was discarded,⁴⁰ but it was conserved by the higher artisans of Christian Europe.

The determination of the area as $\frac{1}{4} \cdot (p \cdot d)$ seems to be inherited, too; whether the variant $A = (\frac{1}{2}d) \cdot (\frac{1}{2}p)$ is also part of the Near Eastern legacy or derived from Archimedes is undecidable.⁴¹ Equally undecidable is the question whether the idea of determining the area from the perimeter was inherited from the Old Babylonian orbit – however much the idea seems unfamiliar to us, the practical problem of finding the volume of a standing log or column would easily call for independent „invention”. So much is sure, however, that none of these formulae in *Geometrical/AC* and *Geometrical/S* are borrowed from Hero’s *Metrica*, and that the treatment of the circle is strongly eclectic in either *Geometrica*;⁴² among other sources they may *also* have used *Metrica*; but material from *Metrica*, if at all there, is by no means privileged.⁴³ As we see, mss AC – in other respects more „Greek” than ms S – even favour the primacy of the perimeter. Particu-

39 The formula $A = (d^2 - \frac{1}{4}d^2) + \frac{1}{21}(d^2 - \frac{1}{4}d^2)$ looks like a similarly corrected version of $A = (d^2 - \frac{1}{4}d^2)$, a formula that seems to be presupposed in N^{os} 32–33 of the Demotic P. Cairo J.E.89127–30,89137–43 (third century BC). The proofs of the same problems use the formula $A = (\frac{1}{4}p) \cdot d$, where p is found as $3d$. See Richard A. Parker, *Demotic Mathematical Papyri*, Providence & London 1972, 40f. Another corrected version of the Demotic rule seems to be found in S:24.39 (the independent treatise from ms S), $A = \frac{1}{2}d^2 + \frac{1}{4}d^2 + \frac{1}{28}d^2$.

40 Al-Khwārizmī thus multiplies (*daraba*) the diameter directly with the number $3\frac{1}{7}$ (*šalāšah wasubūʿ*)

41 Al-Khwārizmī gives the sketched proof that in every regular polygon „you find the area by multiplying half of the perimeter with half of the diameter of the largest inscribed circle” (trans. Gandz, *cit.* 24). This is evidently not a direct reference to the Archimedean proof; but it may equally well be an argument produced by Al-Khwārizmī himself for an inherited formula and an argument inherited together with the formula and sharpened by the mathematician Al-Khwārizmī.

42 This eclecticism is abundantly confirmed both by the random organization of the text, by the shift between ἐπι ... ποιέω- and πολυπλασιάζω-multiplication even within the same chapter (e.g., chapter 17), and the use of division instead of part-taking, in some passages correlated with πολυπλασιάζω-multiplication (e.g., 17.32–36), in others not (e.g., 17.18–22).

43 Even isolated borrowings into ms S and mss AC, chapter 17, are highly unlikely (AC 21.1–3, on the other hand, are clearly derived from Hero but also presented as an extraneous insertion – see note 36): in all cases where formulae are shared with *Metrica*, either the choice of numerical example, the details of the terminology, or both, are different. Since neither *Geometrica* shows traces of stylistic normalization, such differences are significant. Ms A chapter 21, moreover, shows to which extent even expanding rewriting would conserve the original vocabulary.

larly intriguing is the supposedly „Euclidean” formula from AC 17.5. It corresponds so closely to the Arabic formula $A = d^2 - 1/7d^2 - 1/2 \cdot 1/7d^2$ that we may presume it a translation from a language using ascending continued fractions as a standard idiom – either the Arabic or the Aramaic/Syriac.⁴⁴ Since the same formula is also used in the separate treatise S:24, in a context that does not suggest a late Byzantine borrowing of Arabic material, it seems that the Archimedean π had already reached the Syrian practitioners well before the Hegira.⁴⁵

Semicircles offer the same picture – there is no need to go into details. The very fact that they are not dealt with on their own in *Metrica* but are so in both *Geometrica* (and in the brief treatment of mensuration in *De re rustica*) shows that Hero is not the main reference of these works.

The treatment of segments is informative in a different way. As we know from Hero, the determination of their area by means of approximate formulae was old in his world – whatever this world was.⁴⁶ One was pre-, another post-

44 Alternatively, one might think of an adoption of the Archimedean result in Demotic Egypt. On one hand, however, better candidates for a Demotic-Archimedean formula are at hand – see note 39; on the other, the Arabic formula, which would then be likely to derive from the same source, is found in treatises that contain no hints of such a connection.

Ascending continued fractions are not uncommon in *Geometrica/AC*, it is true, but only as the outcome of multiplications and where reduction would yield an unhandy result – e.g., when $7^{1/2} / 10^{1/15} / 75$ is reduced no further than $7 + (3 + 2/5) / 5$ (12.48) or $24 / 49$ expressed as $(3 + 3/7) / 7$ (12.54). But all these cases conserve that form of the ascending continued fraction which is reduced to a mere allusion in the expression $1/7 + 1/14$, and use as their elementary building blocks expressions no less complex than $3/14$.

45 In this connection it should be noted that the circular area is also determined as $A = d^2 - 1/7d^2 - 1/2 \cdot 1/7d^2$ in the Hebrew *Mišnat ha-middot* (ed. Gandz, *cit.* 24). The date of this work is disputed – c. 150 CE according to Gandz, early Islamic period if we believe Gad Sarfatti, *Mathematical Terminology in Hebrew Scientific Literature of the Middle Ages*. [In Hebrew, English abstract]. Jerusalem 1968. As far as I can judge from Sarfatti’s abstract, he overlooks the Aramaic/Syriac influence on the formation on both Hebrew and Arabic terminologies, and that shared metaphors as close at hand as the „arrow” (for the height of a circular segment understood as a „bow-shape”) are likely to be common heritage. All in all, Gandz’s dating seems the better. The subject-matter points in the same direction: *Mišnat ha-middot* gives the post-Archimedean approximation for the circular segment, which is absent from all the Arabic treatises – in particular from al-Khwārizmī’s, which one way or the other is closely related to the *Mišnat*. Cf. also note 48.

46 Segment computations are indeed found in a couple of Old Babylonian texts from a text group which also gives the area of the semicircle as $1/4(d \cdot p)$; unfortunately one is corrupt and the other badly broken, but the former seems to involve arc, chord and height, while the latter has a vague similarity with the formula of „the ancients” but

Archimedean. The mathematical quality of the formulae is not impressive. If Hero is responsible for the Archimedean approximation to small segments, his work was messy (see text before note 32); if, against my expectations, the insertion is later, he was also a not very critical compiler. Both *Geometrica* versions, in any case, are such compilations, and not with *Metrica* as their main source; both also offer approximate formulae for the length of the arc.

The treatment of segments in Arabic treatises is totally different. They find the area from height, chord and arc, using thus a procedure that is both correct and exact, but evidently overdetermined.⁴⁷ This contrast is important: since practitioners' mathematics is invisible, we have almost no other evidence that those of the Hellenistic world (or some of them) had developed canons and procedures of their own – procedures which went no further in the Near Eastern tradition than *Mišnat ha-middot*, in contrast to *Elements* II.12 and the Heronic formula.⁴⁸ The problem of Hero's historical position cannot be reduced to a simple dilemma between a Euclidean-Archimedean and a Near Eastern practitioners' tradition.

Numerical values

Numbers are treacherous – apparently striking coincidences may result from the compulsion of mathematical relations. That the sides of the standard scalene triangle are 13–14–15 may look as evidence for connections, but proves nothing in environments that know the Pythagorean theorem and want to work in integer numbers. Whether the special status of 7 is a number-psychological universal in cultures that count until 7 or is the outcome of cultural diffusion (and, in the latter case, diffusion at which level) is so far undecided. If we want to show that a particular number is chosen because it belongs to a particular problem type and not because the author likes it in general or because it is more conveni-

apparently no more. One text, finally, demonstrates that the Babylonians knew how to calculate the diameter of the circle from the chord and the height of a segment – essential if a major segment is to be found from the formula for the minor segment, or vice versa.

- 47 The reason for this is not ignorance but practical wisdom; al-Karajī teaches indeed how to find the arcs for 60° and 120° and how to interpolate from these values; but he also tells that direct measurement is much better – ed. Hochheim, II, 25f. (*cit.* note 22).
- 48 The only analogous case which I know of is the survival of the „thrice the diameter plus an extra seventh” in Europe, a formula that is not found in Arabic treatises (see note 38). Even in this case, *Mišnat ha-middot* contains the information that „the people of the world [or, „the landmeasurers”] say that the circumference of a circle contains three times and a seventh of the thread” (V.4, trans. Gandz).

ent than other choices, we need to demonstrate that it is *only* chosen in this particular situation and not elsewhere.

The use of 10 as the preferred side length for regular polygons *can* be argued in this way to belong to the situation. We have encountered it as the standard side in the problem on the four sides and the area, but the evidence is much more impressive:

Abū Bakr, *Liber mensurationum*

Square, equilateral triangle

Al-Khwārizmī, *Algebra*, and al-Karajī, *Kāfi*

Equilateral triangle (and the square side corresponding to the paradigmatic irrational root, $\sqrt{200}$)

Ibn Thabāt, *Ghunyah al-Hussāb*

Square, equilateral triangle, regular hexagon (exemplar for regular polygons)

Savasorda, *Liber embadorum*

Square, equilateral triangle

Hero, *Metrica*

First square; regular n -gons, $2 < n < 13$; diameter of first circle

Geometrical/AC

First equilateral triangle; regular n -gons, $6 < n < 13$; second example for $n = 5$.

That 10 is not chosen with this high frequency just because it is a round number, in Greek as well as Hebrew and Arabic, follows if we look at other situations where parameters are chosen freely – e.g. the width of the rectangle that is squeezed into the 13-14-15-triangle in order to produce a convenient trapezium. This width is *never* 10. As far as the Old Babylonian problem is concerned one may notice that this problem is the *only* problem in the whole Old Babylonian record concerned with a single square where the side is 10. The Old Babylonian school norm – evidently connected to the characteristics of the place value system – is 30, mostly in the order of magnitude of „minutes”. Even in *Metrica* and *Geometrical/AC*, 10 possesses no generally favoured status. That it plays a central role precisely as the standard side of the regular polygons is thus an indubitable and very direct reference to the Near Eastern tradition. How it was mediated through local traditions we cannot say.

Phraseology

As far as phraseology is concerned, I shall limit myself to two observations with different but not contradictory implications.⁴⁹

Above, the phrase καθολικῶς /“in general” and its correspondence with the use of *semper* in Abū Bakr’s *Liber mensurationum* was mentioned. In *Geometrical/AC* 21.9, καθολικῶς and ἀεί/“always” occur in the riddle about the circular diameter+perimeter+area, ἀεί without καθολικῶς in a few other circle and segment formulae and in some formulae dealing with regular polygons – and as far as I have noticed nowhere else in AC;⁵⁰ both words tell that a certain number used in the procedure does not depend on the particular free parameter of the problem – in the quadratic problems thus the sum (the coefficients are *not* free, as it follows, they have to be the „natural” parameters of the problem). In the context of mss AC it is thus evident that the problem $d+p+A = \Sigma$ – found for the first time in an Old Babylonian text from a group which gives the semicircular area as $\frac{1}{4}(p \cdot d)$ – is brought into the main text from outside (cf. also note 36 on the status of chapter 21).

S:24 uses καθολικῶς in the square area+perimeter- as well as the diameter +perimeter+ area-problem (S.24.3 and S.24.46–47, respectively), alternating in the latter with πάντοτε/“at all times”; S:24.46 and 47, fully parallel apart from different values of the sum Σ , use both words but in changing places, treating them thus as synonyms. πάντοτε is also used in other problems dealing with circles (N^{os} 43–45, the first of which asks for the separation of $p+d$), while a sequence of problems asking for the separation of area and perimeter of a Pythagorean triangle (N^{os} 10–13) use παντός/“always” in the same function (once πάντοτε). πάντοτε is also used in S:22.4, the formulation in general terms of the $\frac{1}{2}$ -base-height-rule for the triangular area, and together with καθολικῶς in *Geometrical/S* 17.5–6, 19.6 (circle and segment formulae).

Most of the occurrences of the explicitation of general validity are thus within problems that ask explicitly for the „separation” of a sum,⁵¹ a familiar phrase from the Near Eastern tradition already reflected in the Old Babylonian school texts. Even though these texts contain no similar term for generality,

49 But see also note 42 on the conclusions that can be drawn from the distribution of the terms for multiplication.

50 Of course, formulae for circles and their parts and for regular polygons are precisely those where fixed numerical parameters occur; but only a small minority of the corresponding problems contain the ἀεί.

51 And all separation problems in the material except *Geometrical/AC* 17.9 (circular diameter+perimeter) use καθολικῶς, ἀεί, πάντοτε or παντός. The coupling is certainly strong.

there is little doubt that the usage was adopted together with the area+perimeter and diameter+perimeter+area problems, indubitable legacy of the Near Eastern tradition.

As we see, the word chosen to express general validity varies from one text to the other; so does the term for separation. *Geometrical/AC* 21.9 has διαστέλλω (with καθολικῶς and αἰεί); S:24.3 (square area+perimeter) uses διαχωρίζω (with καθολικῶς); S:24.10–13,43–47 (triangular area+perimeter, circle problems including $d+p$ and $d+p+A$) employ ἀποδιαστέλλω (with παντός, πάντοτε and, in the $d+p+A$ -problem, καθολικῶς). The correlations seem to reflect separate developments that were accidentally brought together again, first by Byzantine compilers and next by Heiberg.

This inference is confirmed by other terminological considerations. Many of the παντός/πάντοτε-problems of S:24 (but not S:24-problems in general) share the characteristic phrase φανερόν, „obviously”. The two mixed second-degree problems of S:24 (square area+perimeter, circular diameter+perimeter+area), on the other hand, use quite different terminologies in spite of their shared ultimate origin – the former thus the geometrically suggestive ποιέω ἐπί/“I put on top of” for squaring, the latter πολυπλασιάζω/“I make multiple”, with its purely arithmetical (and even integer-arithmetical) connotations. If we think of Moritz Cantor’s old metaphor,⁵² according to which the development of mathematics is to be likened to a river landscape, the river that had sprung from Near Eastern geometrical practice had dissolved itself in later antiquity into a delta, in a multitude of independent streams now running together, now splitting apart. Hero knew some of them and used them – at times literally – in *Metrica*; *Geometrical/AC* collected others, *Geometrical/S* and S:24 still others. Further studies of terminology and style may help us sort out more details; given the complexity of the situation and the paucity of sources for precisely the practitioners’ level of mathematical activity, however, we are not likely to get very far.

52 Moritz Cantor, *Die römischen Agrimensoren und ihre Stellung in der Geschichte der Feldmesskunst. Eine historisch-mathematische Untersuchung*, 2. (Leipzig 1875).